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Moduli spaces of discrete gravity
I. A few points

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Abstract

Spectral triples describe and generalize Riemannian spin geometries by converting the geometrical information into algebraic data, which consist of an algebra A, a Hilbert space H carrying a representation of A and the Dirac operator D (a selfadjoint operator acting on H). The gravitational action is described by the trace of a suitable function of D.

In this paper we examine the (path-integral)-quantization of such a system given by a finite-dimensional commutative algebra. It is then (in concrete examples) possible to construct the moduli space of the theory, i.e. to divide the space of all Dirac operators by the action of the diffeomorphism group, and to construct an invariant measure on this space.

We discuss expectation values of various observables and demonstrate some interesting effects such as the effect of coupling the system to fermions (which renders finite quantities in cases, where the bosons alone would give infinite quantities) or the striking effect of spontaneous breaking of spectral invariance.

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1. Introduction

Locally all manifolds look just like $\mathbb{R}^d$ and, within the present experimental accuracy, spacetime looks the same. “But now it seems that the empirical notions, on which the measurement of distances is based, namely the notion of a rigid body and a light ray, lose

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their validity at unmeasurable small scales. Hence, it is well conceivable, that spacetime does not fulfill the suppositions of this (Riemannian) geometry, and this one should in fact assume, if it would explain observations more naturally.” As has been pointed out by Riemann 1854 in his famous habilitation thesis.

Nowadays, it seems to be a common belief among theoretical physicists that spacetime possesses (at small distances) a structure different from that of a classical manifold. There are, for instance, many heuristic arguments (e.g. [1], or the ones cited in [2]) for some uncertainty relation of the type $[x^\mu, x^\nu] \neq 0$ caused by quantum effects in the process of measuring a spacetime region with a certain accuracy. Another appealing hint to a noncommutativity of spacetime—visible at high energies—is Chamseddines’ and Connes’ [3] description of the standard model of particle physics as a part of the gravitational field over an appropriately chosen noncommutative space. Other approaches [4–6] formulate the standard model as gauge theory over (essentially) the same noncommutative spacetime, which is given by the tensor product of the algebra of functions over spacetime with some finite-dimensional C^*-algebra.

In this paper we shall restrict ourselves to Connes’ approach, which describes the geometrical structure of spacetime, and especially its spin structure, as a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ [7–10]. The gravitational action can then, for instance, be reformulated by using the “spectral action principle” introduced in [3] as the trace of a suitable function of the Dirac operator D. Hence the gravitational action is directly expressed as a sum over diffeomorphism invariant quantities, namely the eigenvalues of the Dirac operator. Such an action is, of course, invariant not only under diffeomorphisms of spacetime, but under *all* unitaries on the Hilbert space $\mathcal{H}$. That is, however, too much symmetry to simply reproduce the Einstein–Hilbert action of general relativity, since this action is invariant under nothing but the group of diffeomorphisms. The reason for this increase of symmetry is the fact that the Connes–Chamseddine action does not take into account boundary conditions for the metric (i.e. initial data). It only reproduces the integral over the *entire* spacetime of the cosmological constant, the scalar curvature (and possibly higher order terms) [11]. In fact, the eigenvalues of D do not form a complete set of observables for the gravitational field (“one cannot hear the shape of drum”). We refer to [12] for an instructive argument.

Finally, it should also be noted that this description only works for spaces with Euclidean signature of the metric. The generalization to Lorentzian or arbitrary signature is in progress [13–15]. The approach initiated in [15] is actually designed to facilitate the incorporation of initial conditions for the metric in the theory.

Ignoring these open problems in the following, we shall focus our attention on the “established” form of spectral triples with strict Euclidean metric. The spectral action principle then defines an action for the metric corresponding to spectral triples, that possesses all the required properties. It should be stressed that this theory—despite its quantum like appearance—is completely classical, and this raises the problem of its quantization.

No need to say that this would be a rather hard task for generic spectral triples. However, for finite spectral triples, for which the Hilbert space $\mathcal{H}$ is finite-dimensional, one can hope to undertake this job and thereby gain some new insights that might be helpful also for the generic case. Apart from that, the study of finite-dimensional spectral quantum gravity might also be interesting by itself, as it gives rise to very unusual matrix models, with unusual symmetries and a completely new “physical” interpretation.

In [16] the canonical quantization of a particular example has been performed by making use of the observation that there exists a canonical symplectic form on the space of Dirac operators. That idea, however, only works as long as one ignores the (spectral or at least diffeomorphism) invariance of the system, since only then the configuration space is simply given as the space of all Dirac operators. In this series of papers we shall therefore address the problem of correct treatment of the diffeomorphism invariance of spectral actions.

At first thought, it seems to be quite easy to formulate a path integral for a spectral invariant system: denoting the independent eigenvalues of the Dirac operator by $\lambda_1, \ldots, \lambda_n$, any spectral invariant measure can be written in the form

$$\mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_n \rho(\lambda_1, \ldots, \lambda_n)$$

with a density ρ that is symmetric under permutations of its arguments. (There do always exist unitaries which interchange the eigenspaces of D.) Unfortunately it is usually inconvenient to express meaningful observables in terms of the eigenvalues of D, and in fact, even for finite-dimensional spectral triples some observables cannot be written that way. For instance, for the two-point space that we consider in this first paper, the distance of the two points is invariant under the group of diffeomorphisms—which of course is not a very impressing property, as there is only one nontrivial diffeomorphism: the interchange of the two points—and hence an observable. We shall present an example where it is functionally independent of the eigenvalues of the Dirac operator.

So (for such examples) one would only seek for a diffeomorphism invariant measure and try to write it using the entries of D. In [17,18] a classification of finite-dimensional spectral triples has been given, which includes an explicit characterization of the space of all Dirac operators or, more precisely, of those entries which do not necessarily vanish. There did remain, however, a freedom of choice of basis in $\mathcal{H}$, which allows to restrict the range of some entries or even transform them to zero.

Two spectral triples are called unitarily equivalent if they can be transformed into each other by such a choice of basis in $\mathcal{H}$, i.e. if all the data of the triples can be transformed into each other by the same unitary transformation U. Apart from the Dirac operator, it is not hard to classify all data of real, even, finite-dimensional spectral triples—i.e. the representations π, π^{o} of the algebra $\mathcal{A}$, the grading Γ and the reality structure J—up to unitary equivalence. That has been done in [17,18], we shall shortly review the results in Section 2.

But then there still exist unitaries on $\mathcal{H}$, which do commute with π, π^{o}, Γ and J but not with the Dirac operator. Presently it is unfortunately not possible to systematically investigate the consequences of this additional freedom for the classification of finite-dimensional spectral triples *up to unitary equivalence*. For particular examples one can, however, divide the space of Dirac operators by the action of all (such) unitaries. This will be discussed in Section 4.

In the path integral quantization of spectral triples one should, of course, only sum over unitary equivalence classes, and hence the computation of these moduli spaces is essential for our project. As mentioned above, in this paper we shall concentrate on concrete examples based on the two-point space $\mathcal{A} = \mathbb{C}^2$. We hope that they can serve to illustrate not only the complexity of the problem, but also that its study is not in vain.

Note that the diffeomorphisms of the underlying spacetime, i.e. the automorphisms of $\mathcal{A}$, do not belong to the unitary equivalences that we considered above: if they are actually represented unitarily on $\mathcal{H}$, they do not commute with the representations of $\mathcal{A}$. So, in the generic case, one should in addition divide the space of Dirac operators by the action of the diffeomorphism group. In the $\mathbb{C}^2$-examples which we present in this paper, i.e. however trivial.

Despite their simplicity, our examples show several interesting features. At first view, the most striking effect that we find seems to be the spontaneous breaking of spectral invariance, which is actually rather easily understood.

Suppose, for simplicity, that the classical action is given as $S = \mathrm{Tr}\, P(D^2)$ where $P(x)$ is a polynomial that has precisely one minimum x_0 on the real line. Then the minimum of S is given for $D = x_0\,\mathrm{id}$. (In fact, a ground state that is invariant under all unitaries must be of this form.) Now, as explained above, the measure

$$\mathrm{d}\mu = \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_n \prod_{i<j} (\lambda_i^2 - \lambda_j^2)^2\, \mathrm{e}^{-S}$$

is spectral invariant. But then the vacuum expectation value

$$\langle \lambda_k^2 - \lambda_l^2 \rangle = \int \mathrm{d}\mu (\lambda_k^2 - \lambda_l^2)$$

would not vanish in general. That is to say, the unitaries cannot be represented on the Hilbert space of the quantum theory in such a way that the vacuum state would be invariant. (Otherwise the expectation values of any two eigenvalues of D—viewed as observables—would be equal.) As we shall see in the examples, measures like the one above appear rather naturally due to the curved geometry of the moduli space.

Nevertheless, the loss of spectral invariance can obviously be traced back to properties of the chosen measure, and one might well (and should) ask whether such a choice is justified. Indeed, one might as well take the requirement of spectral invariance of the vacuum as a criterion to prefer some measures over others.

On the other hand, it should always be kept in mind that the underlying "manifold" is zero-dimensional and accordingly there is no time and hence no canonical action of the symplectic group. So there is no canonical measure for the path integral, since for (finite-dimensional) quantum mechanical systems the measure is singled out by its invariance under canonical transformations combined with the given classical limit of the system. To be more precise, the absence of canonical transformations should therefore be viewed as ambiguity of the definition of the "classical" action as one might always redefine

$$\mathrm{d}\mu\, \mathrm{e}^{-S} = \mathrm{d}\tilde{\mu}\, \mathrm{e}^{-\tilde{S}}.$$

In fact, writing the above measure in the form $\mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_n\, \mathrm{e}^{-\tilde{S}}$, one sees that the so defined action $\tilde{S}$ does not have a unique minimum. The ground state is degenerate with a nontrivial action of the unitaries via permutations of the various minima, thus leading to a spontaneous breaking of spectral invariance in the quantized theory.

This then points to the most important question we plan to investigate in this project:

> Can one define a classical limit for such systems?

There need not exist an (unique) answer to this question, but if there is one, then it can only be found by a detailed analysis of many examples. In the complementary project explained in

[19,20] the mathematical structure of a perturbative treatment of a measure $\mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_n\, \mathrm{e}^{-S}$ is analyzed in great depth by exploiting the noncommutativity of the underlying manifold.

In this illustrative paper we shall mainly consider Gaussian measures $S \sim \mathrm{Tr}\, D^2$ on the moduli spaces. We should stress once again that one cannot always choose a spectral invariant measure, as there are sometimes observables which cannot be expressed in terms of the eigenvalues of D.

For the two-point space the only interesting observable is the distance d of the points, which in the simplest example, when D has only one (independent) eigenvalue λ, is given by

$$d = \frac{1}{\lambda}.$$

Thus, for the Gaussian measure its vacuum expectation value

$$\langle d \rangle = \int_0^\infty \mathrm{d}\lambda \frac{1}{\lambda}\, \mathrm{e}^{-t\lambda^2} = \infty$$

diverges, as had to be expected, since this corresponds to the unique "classical" ground state $D = 0$. However, in other examples, where D has more than one eigenvalue, we find finite vacuum expectation values for d, pointing to some attractive force between the points caused by quantum effects.

Another way to achieve finite expectation values of d is to couple the system to fermions. Then as

$$\int_0^\infty \mathrm{d}\lambda\, \mathrm{e}^{\langle \psi, D\psi \rangle} \sim \lambda^2,$$

one has

$$\langle d \rangle_F \sim \int_0^\infty \mathrm{d}\lambda \lambda\, \mathrm{e}^{-t\lambda^2} < \infty.$$

As mentioned above this paper does, however, not aim at illustrating these (and other) effects, but to work out the problems one is facing when trying to quantize finite spectral triples.

2. Review: finite spectral triples

We start with a very short survey of the main facts on finite (discrete) spectral triples, thereby fixing the notation for the following sections. A detailed version can be found in [17,18] or in [11].

The main ingredients of a spectral tripel are an (involutive) algebra $\mathcal{A}$, a Hilbert space $\mathcal{H}$ that carries a representation of $\mathcal{A}$ and a selfadjoint operator D (Dirac operator) acting on $\mathcal{H}$. We restrict our considerations to complex algebras. In addition one has (in the case of real triples) an antilinear operator J and for even dimensions a grading Γ on $\mathcal{H}$.

In the following the structure of $\mathcal{A}$, $\mathcal{H}$ and D will be analyzed by using the axioms for spectral triples. For finite spectral triples the algebra and the Hilbert space are finite-dimensional and we repeat here only the relevant axioms for that case:

- $\mathcal{A}$ is a C^*-algebra.
- The Hilbert space $\mathcal{H}$ carries a ($*$-) representation π of $\mathcal{A}$.
- D is a selfadjoint operator on $\mathcal{H}$.
- Γ is a symmetry (i.e. $\Gamma = \Gamma^*$, $\Gamma^2 = 1$) that commutes with π and anticommutes with D:

$$[\pi(a), \Gamma] = 0 \quad \forall a \in \mathcal{A}, \qquad \Gamma D + D\Gamma = 0.$$

- J is an antiunitary operator on $\mathcal{H}$, which fulfills $J^2 = 1$.
- The map $\pi^{\mathrm{o}}(a) := J\pi(a^*)J^{-1}$ is a representation of the opposite algebra $\mathcal{A}^{\mathrm{o}}$ that commutes with π

$$\pi^{\mathrm{o}}(a)\pi^{\mathrm{o}}(b) = \pi^{\mathrm{o}}(ba), \qquad [\pi(a), \pi^{\mathrm{o}}(b)] = 0 \quad \forall a, b \in \mathcal{A}.$$

- "Order one condition":

$$[[D, \pi(a)], \pi^{\mathrm{o}}(b)] = 0 \quad \forall a, b \in \mathcal{A}.$$

- "Orientability":

$$\exists c \in \mathcal{A} \otimes \mathcal{A}^{\mathrm{o}} \quad \text{with } \pi(c) = \Gamma.$$

  The representation of $\mathcal{A} \otimes \mathcal{A}^{\mathrm{o}}$ is defined by $\pi(a \otimes b) := \pi(a)\pi^{\mathrm{o}}(b)$.
- "Poincaré-duality":
  The Fredholm index of the operator D defines a nondegenerate intersection form on $K_\bullet(A) \times K_\bullet(A)$.
- The following relations hold:

$$[J, D] = 0, \qquad [J, \Gamma] = 0.$$

One can then characterize the solutions $\mathcal{A}$, $\mathcal{H}$ and D of these conditions. First of all, each finite-dimensional complex C^*-algebra is a direct sum of matrix algebras

$$\mathcal{A} = \oplus_{i=1}^{k} M_{n_i}(\mathbb{C}).$$

To analyze the structure of the Hilbert space we decompose it in the following way: define P_i as the projector on the ith subalgebra of $\mathcal{A}$

$$P_i := 0_{n_1 \times n_1} \oplus \cdots \oplus \mathbf{1}_{n_i \times n_i} \oplus 0 \oplus \cdots \oplus 0$$

and set

$$\mathcal{H}_{ij} := \pi(P_i)\pi^{\mathrm{o}}(P_j)\mathcal{H}, \tag{1}$$

where π and π^{o} are the representations of $\mathcal{A}$ and $\mathcal{A}^{\mathrm{o}}$, respectively. Up to unitary equivalence, the only irreducible representation of $M_n(\mathbb{C})$ is $\mathbb{C}^n$, so each $\mathcal{H}_{ij}$ has the form

$$\mathcal{H}_{ij} = \mathbb{C}^{n_i} \otimes \mathbb{C}^{r_{ij}} \otimes \mathbb{C}^{n_j}, \tag{2}$$

where the representation π acts on the left factor in the tensor product and π^{o} on the right.

From the orientability condition it then follows that the grading Γ acts on the Hilbert subspaces $\mathcal{H}_{ij}$ only as $\pm$id. If we define this sign as γ_{ij} and moreover $q_{ij} := \gamma_{ij} r_{ij}$ the full information about the Hilbert space $\mathcal{H}$ and the grading Γ is encoded in the matrix q.

The reality operator J maps $\mathcal{H}_{ij}$ to $\mathcal{H}_{ji}$ and from its invertibility it then follows that the matrix q is symmetric. This matrix turns out to coincide with the intersection form in K-theory defined by D (the axiom of Poincaré-duality then requires q to be invertible).

For our project, the only interesting part of a spectral triple is the Dirac operator D. The axioms are strong restrictions on D, which we shall describe in the remainder of this section. With the definitions

$$D_{ij,kl} : \mathcal{H}_{kl} \to \mathcal{H}_{ij}, \tag{3}$$

i.e. $D_{ij,kl} := \pi(P_i) \circ \pi^{\mathrm{o}}(P_j) \circ D \circ \pi(P_k) \circ \pi^{\mathrm{o}}(P_l)$ and

$$a_i := aP_i = P_i a,$$

one can prove the following relations.

Theorem 2.1 (Structure of D).

- $D_{ij,kl} = D^*_{kl,ij}$;
- $D_{ij,kl} = 0$ if $\gamma_{ij} = \gamma_{kl}$;
- $[D_{ij,il}, \pi(a_i)] = 0\,\forall a_i$;
- $[D_{ij,kj}, \pi^{\mathrm{o}}(b_j)] = 0\,\forall b_j$;
- $D_{ij,kl} = 0$ if $i \neq k$ and $j \neq l$;
- $D_{ij,ij} = 0$.

Moreover, using the freedom in the choice of basis in $\mathbb{C}^{r_{ij}}$ *it can be shown, that one can always choose a basis of* $\mathcal{H}$ *with the following properties*:

- $\{v_1, \dots, v_n\}$ *is a basis for* $\mathcal{H}_{ij}$;
- $\{w_1, \dots, w_n\}$ *is a basis for* $\mathcal{H}_{ji}$;
- $Jv_k = w_k$ *if* $i \neq j$;
- $Jv_k = J(x \otimes y \otimes z) = \bar{z} \otimes \bar{y} \otimes \bar{x}$ *if* $i = j$.

(*As the proofs of this fact in the literature are actually not completely compelling, we present a new proof in* Appendix A.)

Then D *has the additional properties*

- $D_{ij,kl} = \bar{D}_{ji,lk}$ *for* $i \neq j$ *and* $k \neq l$;
- $D_{ii,kl} = J \circ D_{ii,lk} \circ J$;
- $D_{ij,kk} = J \circ D_{ji,kk} \circ J$.

In conclusion, given the algebra $\mathcal{A}$ and the intersection form q the dimensions of the Hilbert spaces in the composition (1) are fixed and the structure of all possible Dirac operators is given by Theorem 2.1. As mentioned in Section 1, this result does, however, not provide a complete classification of finite spectral triples *up to unitary equivalence*:

While the freedom of choice for the basis of $\mathcal{H}$ has been used to bring $\pi(\mathcal{A})$, Γ and J into a canonical form, it has not been investigated, whether the remaining freedom—i.e. the unitaries which commute with $\pi(\mathcal{A})$, Γ and J—can be used to restrict the space of Dirac operators. We shall return to this question in Section 4.

3. Dirac operators and distances in the two-point space

Before we turn our attention to the quantization of finite spectral triples, we shall work out the most important observable for the two-point space $\mathcal{A} = \mathbb{C}^2$, namely the distance of the two points. Other important observables are the eigenvalues of the Dirac operator, of course. How to obtain their analytic expression in terms of the matrix entries of D is clear, however.

For a general commutative spectral triple, with $\mathcal{A} = C^\infty(M)$ the geodesic distance of two points in M can be recovered from the purely algebraic data using Connes' celebrated distance formula

$$d(p, q) = \sup_{f\in C^\infty(M)}\{|f(p) - f(q)| : \|[D, f]\| \leq 1\}.$$

(The Hilbert space $\mathcal{H}$ where D acts on is the space of square integrable spinors on M and the representation of a function $f \in C^\infty(M)$ is given by pointwise multiplication.)

This formula remains valid in case the algebra $\mathcal{A}$ is not longer commutative, if one replaces the "points" by the pure states of the algebra. (This corresponds to viewing $f(p)$ as $\delta_p(f)$ in the commutative case.) In the general case we write for pure states ϕ, ψ:

$$d(\phi, \psi) = \sup_{a\in A}\{|\phi(a) - \psi(a)| : \|[D, a]\| \leq 1\}.$$

In the following we compute the distance-formula for the two-point space, $A = \mathbb{C} \oplus \mathbb{C}$. We hope that this example also illustrates the formalism for finite spectral triples given in the previous section. There it was shown that spectral triples for this algebra are determined by (invertible) 2×2 matrices with integer entries. If one takes into account the symmetry of q, then the most general matrix is

$$q = \begin{pmatrix} k & -l \\ -l & m \end{pmatrix}, \quad k, l, m \in \mathbb{N}.$$

Here $\mathcal{H}$, π and D are given by

$$\mathcal{H} = \mathcal{H}_{11} \oplus \mathcal{H}_{12} \oplus \mathcal{H}_{21} \oplus \mathcal{H}_{22} \cong \mathbb{C}^k \oplus \mathbb{C}^l \oplus \mathbb{C}^l \oplus \mathbb{C}^m,$$

$$\pi(x, y) = \begin{pmatrix} x\mathbf{1}_k & 0 & 0 & 0 \\ 0 & x\mathbf{1}_l & 0 & 0 \\ 0 & 0 & y\mathbf{1}_l & 0 \\ 0 & 0 & 0 & y\mathbf{1}_m \end{pmatrix},$$

$$D = \begin{pmatrix} 0 & D_{11,12} & D_{11,21} & 0 \\ D_{12,11} & 0 & 0 & D_{12,22} \\ D_{21,11} & 0 & 0 & D_{21,22} \\ 0 & D_{22,12} & D_{22,21} & 0 \end{pmatrix}.$$

Because of the symmetries of D (Theorem 2.1) this can be simplified to

$$D_{11,12} = \bar{D}_{11,21} = D^*_{12,11} = D^{\mathrm{T}}_{21,11} =: M : \mathbb{C}^l \to \mathbb{C}^k,$$
$$D_{12,22} = \bar{D}_{21,22} = D^*_{22,12} = D^{\mathrm{T}}_{22,21} =: N : \mathbb{C}^m \to \mathbb{C}^l.$$

Thus one gets

$$D = \begin{pmatrix} 0 & M & \bar{M} & 0 \\ M^* & 0 & 0 & N \\ M^{\mathrm{T}} & 0 & 0 & \bar{N} \\ 0 & N^* & N^T & 0 \end{pmatrix},$$

and with this notation one then obtains by a straightforward calculation

$$[D, \pi(x, y)] = (x - y) \begin{pmatrix} 0 & 0 & -\bar{M} & 0 \\ 0 & 0 & 0 & -N \\ M^{\mathrm{T}} & 0 & 0 & 0 \\ 0 & N^* & 0 & 0 \end{pmatrix} =: (x - y)R.$$

The norm of this one form $[D, \pi(x, y)]$ that we shall need for the distance is then conveniently calculated as

$$\|R\|^2 = \|R^*R\| = \left\| \begin{pmatrix} \bar{M}M^{\mathrm{T}} & 0 & 0 & 0 \\ 0 & NN^* & 0 & 0 \\ 0 & 0 & M^{\mathrm{T}}\bar{M} & 0 \\ 0 & 0 & 0 & N^*N \end{pmatrix} \right\|$$
$$= \max\{\|\bar{M}M^{\mathrm{T}}\|, \|NN^*\|, \|M^{\mathrm{T}}\bar{M}\|, \|N^*N\|\}.$$

But $\|\bar{M}M^{\mathrm{T}}\| = \|MM^*\|$ (conjugation of the entries does not change the norm) and in addition $\|MM^*\| = \|M^*M\|$. For nonquadratic matrices this is not obvious but nevertheless true: MM^* and M^*M are selfadjoint matrices so their norm equals the biggest eigenvalue. However the eigenvalues are equal, though they have a different degree of degeneracy, as one can see from the following consideration: if λ is an eigenvalue of MM^* for the eigenvector v, then M^*v is an eigenvector of M^*M for the same eigenvalue λ. Hence one gets

$$\|R\|^2 = \max\{\|M^*M\|, \|N^*N\|\}$$

and finally

$$\|[D, \pi(x, y)]\| = |x - y|\sqrt{\max(\|M^*M\|, \|N^*N\|)}.$$

As $|x - y|$ is nothing but $|\phi(x, y) - \psi(x, y)|$ for the two pure states on $\mathbb{C}^2$, one immediately obtains

$$d(p_1, p_2) = \sup\{|x - y| : \|[D, \pi(x, y)]\| \le 1\} = [\max(\sqrt{\|M^*M\|}, \sqrt{\|N^*N\|})]^{-1}.$$

It is important to note that this distance can—apart from very particular examples—not be expressed in terms of the eigenvalues of D. This is due to the fact that—as a consequence of the order-one condition—the one-forms $[D, \pi(x, y)]$ have much fewer nonvanishing entries than D. Example 4.4 illustrates this effect of "isospectral two-point spaces".

4. Moduli spaces and quantization

Path integrals are based on the idea that each possible state, which the system can pass on its way from the initial to the final state, contributes to the transition amplitude. In the context of gravity one would have to find a way to sum over Lorentzian manifolds, a problem that is not yet solved in $(3+1)$ dimensions. In our framework of finite geometries, however, it is straightforward to define such a summation in certain examples.

For a given spectral triple $(\mathcal{A}, q)$ we want to define a state sum

$$\mathcal{Z} = \mathcal{N} \int \mathcal{D}D\, \mathrm{e}^{-S(D)},$$

where the curly D denotes the invariant measure that we are hunting for.

In order to do this we first need to classify equivalent spectral triples for a given algebra $\mathcal{A}$ and given intersection form q. Let us briefly recall the definition.

Definition 4.1. Two spectral triples $(\mathcal{A}, \mathcal{H}, \pi, D, \Gamma, J)$ and $(\mathcal{A}, \mathcal{H}', \pi', D', \Gamma', J')$ are said to be equivalent, if there is a unitary operator $U : \mathcal{H} \to \mathcal{H}'$ with the properties that the following diagram commutes for $F = \pi(a), J, \Gamma, D$:

$$\begin{array}{ccc} \mathscr{H} & \xrightarrow{U} & \mathscr{H}' \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F'} \\ \mathscr{H} & \xrightarrow{U} & \mathscr{H}' \end{array}$$

Since in our framework the dimensions of the Hilbert spaces are given by the matrix q we can restrict the considerations to the same Hilbert space $\mathcal{H} = \mathcal{H}'$. Moreover the canonical form of J and Γ is achieved by a suitable choice of basis in $\mathcal{H}$. But then, there still exist unitary maps on $\mathcal{H}$ which commute with π, J and Γ. These transformations characterize the equivalence classes of Dirac operators we are looking for. The basic structure of these classes is quite easily described: if U commutes with the representation π (and also with the opposite π^{o}) it fulfills

$$U\mathcal{H}_{ij} = \mathcal{H}_{ij}$$

and is of the form

$$U_{ij} = \mathbf{1}_{n_i} \otimes u_{ij} \otimes \mathbf{1}_{n_j}, \qquad U_{ij} := U|_{\mathcal{H}_{ij}},$$

where now u_{ij} is a unitary map $u_{ij} : \mathbb{C}^{r_{ij}} \to \mathbb{C}^{r_{ij}}$. The relation $[U, J] = 0$ then leads to the further restriction

$$u_{ij} = \bar{u}_{ji}.$$

In particular, the matrices u_{ii} are orthogonal matrices (with real entries). Thus our task consists of finding equivalence classes for the relation

$$D \sim UDU^*, \qquad D_{ij,kl} \sim u_{ij} D_{ij,kl} u^*_{kl},$$

where U is restricted by the explained relations.

Returning to the state sum $\mathcal{Z}$ one should remark, that a "classically" (stable) spectral invariant action $S(D)$ on finite-dimensional Hilbert spaces can be written in the form

$$S(D) = \sum_{k=-\infty}^{\infty} t_k \operatorname{tr} D^{2k}.$$

It is sufficient to sum over even exponents since $\operatorname{tr} D^{2k+1} = 0$ because D anticommutes with the grading Γ. As we stressed in the introduction, one should not take the term "classical action" too seriously, as the absence of time and thus of canonical transformations makes the definition of a classical limit of the system rather difficult.

We shall sometimes also couple the system to fermions $\psi \in \mathcal{H}$, whose classical action is then given as

$$S_{\text{ferm}} = \langle \psi | D | \psi \rangle.$$

We call them fermions as we quantize them according to Fermi–Dirac-statistics, though there is of course no spin-statistics theorem that would tell us to do so. In particular, if we integrate the fermions out, as we shall do throughout this paper, the effective action is given as

$$\mathcal{Z}_F = \mathcal{N}' \int \mathcal{D}D \, \mathrm{e}^{-S_F(D)},$$

where

$$S_F(D) = S(D) - \ln \det D.$$

Note that in most cases $\det D$ vanishes identically, so for a sensible definition of the fermionic action we must calculate $\det D$ on the complement of the kernel of D. The normalization constants $\mathcal{N}$ and $\mathcal{N}'$ will be chosen in such a way that $\mathcal{Z} = 1$ and $\mathcal{Z}_F = 1$ for the Gaussian $S = t/2 \operatorname{tr} D^2$ (that is why the parameter t appears in the normalization constants).

Let us now come to concrete examples, remember that we only treat the algebra $\mathbb{C} \oplus \mathbb{C}$.

Example 4.2.

$$q = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}.$$

The corresponding Hilbert space is $\mathcal{H} = \mathcal{H}_{11} \oplus \mathcal{H}_{12} \oplus \mathcal{H}_{21} \cong \mathbb{C}^3$ and the Dirac operator has the form

$$D = \begin{pmatrix} 0 & \bar{m} & m \\ m & 0 & 0 \\ \bar{m} & 0 & 0 \end{pmatrix}, \quad m \in \mathbb{C}.$$

Now consider a unitary matrix U obeying the above discussed restrictions (leading to an equivalent Dirac operator). It has the form

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & \bar{a} \end{pmatrix}, \quad a \in U(1)$$

and so we have

$$UDU^* = \begin{pmatrix} 0 & \bar{m}\bar{a} & ma \\ ma & 0 & 0 \\ \bar{m}\bar{a} & 0 & 0 \end{pmatrix}.$$

Obviously one can achieve with an appropriate choice of a ($a = \bar{m}/|m|$) that

$$D = \begin{pmatrix} 0 & m & m \\ m & 0 & 0 \\ m & 0 & 0 \end{pmatrix}, \quad m \in \mathbb{R}, m \geq 0,$$

so each class of equivalent Dirac operators has a representative of this form.

D has eigenvalues $0, \pm m\sqrt{2}$ and the kernel of D is spanned by a fixed vector $(0, 1, -1)^{\mathrm{T}}$. Neglecting the kernel one gets for the determinant

$$\det D = -2m^2.$$

Since D has only one independent eigenvalue $\lambda = m\sqrt{2}$ the invariant measure is clear

$$\mathcal{Z} = \mathcal{N} \int_0^\infty \mathrm{d}\lambda\, \mathrm{e}^{-S(\lambda)},$$

respectively (remember $\det D = -2m^2 = -\lambda^2$)

$$\mathcal{Z}_F = -\mathcal{N}' \int_0^\infty \mathrm{d}\lambda \lambda^2\, \mathrm{e}^{-S(\lambda)}.$$

Fixing the constants $\mathcal{N}$ and $\mathcal{N}'$ we get

$$\mathcal{Z} = 2\sqrt{\frac{t}{\pi}} \int_0^\infty \mathrm{d}\lambda\, \mathrm{e}^{-S(\lambda)}, \qquad \mathcal{Z}_F = 4\sqrt{\frac{t^3}{\pi}} \int_0^\infty \mathrm{d}\lambda \lambda^2\, \mathrm{e}^{-S(\lambda)}.$$

In Section 3 we calculated for the distance $d(1,2) = 1/m = \sqrt{2}/\lambda$, so for the vacuum expectation values we end up with the following expressions:

$$\langle d(1,2)\rangle = 4\sqrt{\frac{t}{2\pi}}\int_0^\infty \mathrm{d}\lambda \frac{1}{\lambda}\,\mathrm{e}^{-S(\lambda)}, \qquad \langle d(1,2)\rangle_F = 8\sqrt{\frac{t^3}{2\pi}}\int_0^\infty \mathrm{d}\lambda\lambda\,\mathrm{e}^{-S(\lambda)}.$$

That leads us to a first interesting observation.

Lemma 4.3. *For each polynomial action* $S(D) = \sum_{k=0}^{n} t_k\lambda^{2k}$ *the vacuum expectation value of the distance is*

$$\langle d(1,2)\rangle = \infty.$$

If we have in addition $t_{-1} \neq 0$, *we get* $\langle d(1,2)\rangle < \infty$. *On the other hand this value remains always finite in the fermionic case*:

$$\langle d(1,2)\rangle_F < \infty.$$

For $S = t\lambda^2$, *we obtain*

$$\langle d(1,2)\rangle_F = 4\sqrt{\frac{t}{2\pi}}.$$

The proof is obvious and we skip it here. This example shows that terms of the form $t_{-1}\lambda^{-2}$ can be used to regularize the vacuum expectation value of the distance. The same effect is due to the coupling of fermions: it leads to an attractive force between the points, which is strong enough to give finite results.

After this brief warm-up, we now come to more sophisticated examples.

Example 4.4. For the intersection form

$$q = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

the Hilbert space is $\mathbb{C}^4$ and the most general Dirac operator is of the form

$$D = \begin{pmatrix} 0 & m & \bar{m} & 0 \\ \bar{m} & 0 & 0 & \mu \\ m & 0 & 0 & \bar{\mu} \\ 0 & \bar{\mu} & \mu & 0 \end{pmatrix}, \qquad m, \mu \in \mathbb{C}.$$

The admissible unitary transformations are

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & \bar{a} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad a \in U(1),$$

so only one complex phase can be eliminated in

$$UDU^* = \begin{pmatrix} 0 & m\bar{a} & \bar{m}a & 0 \\ \bar{m}a & 0 & 0 & \mu a \\ m\bar{a} & 0 & 0 & \bar{\mu}\bar{a} \\ 0 & \bar{\mu}\bar{a} & \mu a & 0 \end{pmatrix}$$

by choosing for example $a = m/|m|$. Hence, in this case the equivalence classes of Dirac operators can be parametrized by a real nonnegative m and a complex number μ.

For the distance one then has

$$d(1,2) = \frac{1}{\max\{m, |\mu|\}},$$

whereas the eigenvalues of the Dirac operator are given by

$$\lambda_\pm^2 = m^2 + |\mu|^2 \pm |m^2 + \mu^2| \tag{4}$$

and thus depend on the phase of μ, respectively, the relative phase of μ and m. The reader then easily verifies, that $d(1,2)$ cannot be expressed as a functional of $\lambda_\pm^2$ as only functions of the combination $\lambda_+^2 + \lambda_-^2$ would be independent of the phase of μ (this was mentioned at the end of Section 3).

Nevertheless the eigenvalues do set certain bounds on the magnitude of $d(1,2)$, e.g. one has the relation:

$$\frac{\sqrt{2}}{|\lambda_+|} \le d(1,2) \le \frac{2}{|\lambda_+|}.$$

Because of the fact, that it is impossible to express the distance in terms of the eigenvalues, we shall only seek for a measure that is invariant under diffeomorphisms—which is trivial here—but not under all unitaries on $\mathcal{H}$.

Such a diffeomorphism invariant state sum for $S(D) = (t/4)\ \mathrm{tr}\ D^2 = t(m^2 + |\mu|^2)$ is then given by

$$\begin{aligned} \mathcal{Z} &= \mathcal{N} \int_0^\infty \mathrm{d}m \int \mathrm{d}\mu\, \mathrm{d}\bar{\mu}\, \mathrm{e}^{-t(m^2+|\mu|^2)} = 2\pi\mathcal{N} \int_0^\infty \mathrm{d}m\, \mathrm{d}r r\, \mathrm{e}^{-t(m^2+r^2)} \\ &= 2\pi\mathcal{N} \int_0^\infty \mathrm{d}m\, \mathrm{d}r\, \mathrm{e}^{-W(m,r)}, \end{aligned}$$

where $W(m,r) := t(m^2 + r^2) - \ln r$. If we use the estimate

$$\sqrt{m^4 + |\mu|^4 + 2m^2|\mu|^2 \cos(2\varphi)} \ge \sqrt{(m^2 - |\mu|^2)^2},$$

we get

$$\begin{aligned} \lambda_-^2 &= m^2 + |\mu|^2 - \sqrt{m^4 + |\mu|^4 + 2m^2|\mu|^2 \cos(2\varphi)} \\ &\le m^2 + |\mu|^2 - \sqrt{(m^2 - |\mu|^2)^2} = 2\min\{m^2, |\mu|^2\} \end{aligned}$$

and thus

$$|\lambda_+| - |\lambda_-| \geq \sqrt{2}|m - |\mu||.$$

Using this result we can then calculate

$$\begin{aligned}\langle|\lambda_+| - |\lambda_-|\rangle \geq \sqrt{2}\langle|m - |\mu||\rangle &= 2\pi\mathcal{N}\sqrt{2}\int_0^\infty \mathrm{d}m\,\mathrm{d}r r|m-r|\mathrm{e}^{-t(m^2+r^2)}\\ &= \frac{2\pi\mathcal{N}\sqrt{2}}{4t^2} > 0\end{aligned}$$

showing that the vacuum expectation value of the two quantized eigenvalues is different. But this should not be the case for a spectral invariant quantum theory, because of the following observation (already discussed in Section 1). If

$$S(D^2) = \sum_i P(\lambda_i^2)$$

is spectral invariant and if the polynomial $P(\lambda)$ has a unique extremum, then all the eigenvalues of D^2 at the extremal point of S are identical (which follows directly from spectral invariance, because the groundstate must be invariant especially under permutations of the eigenvalues).

It follows from the small calculation above that the unitary transformations that correspond to these transformations are not represented in the quantized theory in such a way that the groundstate is invariant. Clearly, this violation of spectral invariance can be traced back to the fact that the measure we used is not spectral invariant, which can be clarified by taking a look at the effective action

$$W(m, r) := t(m^2 + r^2) - \ln r.$$

Its minimum lies at the point $r = 1/\sqrt{2t}$ and not at $r = 0$.

Example 4.5.

$$q = \begin{pmatrix} 2 & -2 \\ -2 & 0 \end{pmatrix}.$$

Here $\mathcal{H} = \mathbb{C}^6$ and a Dirac operator looks like

$$D = \begin{pmatrix} 0 & m & \bar{m} \\ m^* & 0 & 0 \\ m^{\mathrm{T}} & 0 & 0 \end{pmatrix}, \quad m \in \mathbb{C}^{2\times 2}.$$

Unitary equivalence of the corresponding spectral triples is given by transformations of the form

$$U = \begin{pmatrix} A & 0 & 0 \\ 0 & V & 0 \\ 0 & 0 & \bar{V} \end{pmatrix}, \quad A \in O(2), \quad V \in U(2).$$

This leads to

$$UDU^* = \begin{pmatrix} 0 & AmV^* & A\bar{m}V^{\mathrm{T}} \\ Vm^*A^{\mathrm{T}} & 0 & 0 \\ \bar{V}m^{\mathrm{T}}A^{\mathrm{T}} & 0 & 0 \end{pmatrix}$$

and so we have to look for a representative in the class of Dirac operators under the transformation

$$m \to AmV, \quad A \in O(2), \quad V \in U(2).$$

The solution of this problem is given by the following theorem.

Theorem 4.6. *Let m be an arbitrary nonsingular 2×2 matrix. Then there is a unique positive* (*selfadjoint*) *matrix C of the following form:*

$$C = \begin{pmatrix} a & \mathrm{i}c \\ -\mathrm{i}c & b \end{pmatrix}, \quad a, b, c \in \mathbb{R}, \quad a \geq b \geq 0, \quad ab \geq c^2, \tag{5}$$

as well as a unitary matrix U and an orthogonal matrix O (*both unique if $a \neq b$*) *such that*

$$m = OCU.$$

Proof. The technical basis for the proof is the following quick calculation:

$$\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x & z \\ \bar{z} & y \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} = \begin{pmatrix} X & Z \\ \bar{Z} & Y \end{pmatrix}$$

with

$$X = x\cos^2\alpha + y\sin^2\alpha + 2(\operatorname{Re} z)\sin\alpha\cos\alpha, \tag{6}$$

$$Z = (y - x)\sin\alpha\cos\alpha + (\operatorname{Re} z)(\cos^2\alpha - \sin^2\alpha) + \mathrm{i}(\operatorname{Im} z), \tag{7}$$

$$Y = x\sin^2\alpha + y\cos^2\alpha - 2(\operatorname{Re} z)\sin\alpha\cos\alpha \tag{8}$$

In particular the imaginary part of the off diagonal entry is invariant under orthogonal transformations. Now the rest of the proof is straightforward using spectral decomposition of m. □

In the following, we explicitly construct the invariant measure on the space of Dirac operators by using the Faddeev–Popov method:

We know that the equivalence classes are given by $m \sim OmU$, with $O \in O(2)$ and $U \in U(2)$. Moreover for each m there exists a positive T and a unitary U such that $m = TU$, so one can choose a positive representative in each class. The remaining symmetry is the equivalence

$$T \sim OTO^{\mathrm{T}}, \qquad O \in O(2).$$

Now one can use a gauge fixing of the form

$$\mathrm{Re}\, T_{12} = 0$$

and employ the Faddeev–Popov method. The invariant measure on the space of selfadjoint matrices is known to be

$$\mathrm{d}H = \mathrm{d}p\, \mathrm{d}q\, \mathrm{d}\,\mathrm{Re}(z)\, \mathrm{d}\,\mathrm{Im}(z) \quad \text{for} \;\; H = \begin{pmatrix} p & z \\ \bar{z} & q \end{pmatrix}, \qquad p, q \in \mathbb{R}, \;\; z \in \mathbb{C}.$$

Starting from that fact, we calculate the invariant measure on the space of matrices C of the form as in (5), which is given by the well-known formula

$$\mathcal{Z} = \mathcal{N} \int_{H \geq 0} \mathrm{d}H \delta(\mathrm{Re}\, H_{12})\, \Delta_{\mathrm{FP}} \mathrm{e}^{-S(H)},$$

where

$$\Delta_{\mathrm{FP}}^{-1} = \int \mathrm{d}\alpha \delta(\mathrm{Re}\, H_{12}^{\alpha})$$

and H_{12}^{α} is the 12-element after the rotation of H about α which is given by formula (7). The calculation of

$$\Delta_{\mathrm{FP}}^{-1} = \int \mathrm{d}\alpha \delta((q-p)\, \sin\alpha\, \cos\alpha + (\cos^2\alpha - \sin^2\alpha)\, \mathrm{Re}\, z)$$

can then be carried out using the formula

$$\delta(f(x)) = \sum_i \frac{1}{|f'(x_i)|} \delta(x - x_i) \quad \text{summing over the zeros } x_i \text{ of } f.$$

Finally, one gets the (surprisingly simple) result

$$\Delta_{\mathrm{FP}}(H) = \sqrt{(p-q)^2 + 4(Re\, z)^2}.$$

Now writing everything in terms of matrices C, i.e. taking into account positivity and the condition $a \geq b \geq 0$, one finally gets for the invariant integral

$$\int f(C)\, \mathrm{d}C = \int_0^{\infty} \mathrm{d}a \int_0^{a} \mathrm{d}b \int_{-\sqrt{ab}}^{\sqrt{ab}} \mathrm{d}c\, f(C),$$

and for the corresponding state sum—with boundary condition (b.c.) $H \geq 0$, $H_{11} \geq H_{22} \geq 0$

$$\mathcal{Z} = \mathcal{N}\int_{(b.c.)} \mathrm{d}H\delta(\mathrm{Re}\, H_{12})\, \Delta_{\mathrm{FP}}\, \mathrm{e}^{-S(H)}$$
$$= \mathcal{N}\int_0^\infty \mathrm{d}a \int_0^a \mathrm{d}b \int_{-\sqrt{ab}}^{\sqrt{ab}} \mathrm{d}c\sqrt{(a-b)^2}\, \mathrm{e}^{-S(a,b,c)}.$$

One can see from the last expression that (even for the free classical action $S(D) = t_1 \operatorname{tr} D^2$) the strongest contribution does not stem from the configuration $C = 0$ ($\Leftrightarrow D = 0$) and accordingly the vacuum expectation value for the distance remains finite.

The distance in this example is given by the inverse of the biggest eigenvalue of C, i.e.

$$d(1,2) = \left(\frac{a+b}{2} + \sqrt{\left(\frac{a-b}{2}\right)^2 + c^2}\right)^{-1}.$$

For the action

$$S = \tfrac{1}{2} \operatorname{tr} D^2 = 2(a^2 + b^2 + 2c^2),$$

we, therefore, get

$$\langle d(1,2)\rangle = \mathcal{N}\int_0^\infty \mathrm{d}a \int_0^a \mathrm{d}b \int_{-\sqrt{ab}}^{\sqrt{ab}} \mathrm{d}c \frac{(a-b)}{a+b+\sqrt{(a-b)^2+4c^2}}\, \mathrm{e}^{-2(a^2+b^2+2c^2)}.$$

All constants are absorbed in $\mathcal{N}$ from now on. To show finiteness we use the fact that

$$\frac{1}{a+b+\sqrt{(a-b)^2+4c^2}}\, \mathrm{e}^{-2(a^2+b^2+2c^2)} \leq \frac{1}{2a}\, \mathrm{e}^{-2(a^2+b^2)}$$

(remember $a \geq b$ in the integration domain). This leads to

$$\langle d(1,2)\rangle \leq \mathcal{N}\int_0^\infty \mathrm{d}a \int_0^a \mathrm{d}b \int_0^{\sqrt{ab}} \mathrm{d}c \frac{a-b}{a}\, \mathrm{e}^{-2(a^2+b^2)}.$$

The integration over c gives $\sqrt{ab}$, so we have

$$\langle d(1,2)\rangle \leq \mathcal{N}\int_0^\infty \mathrm{d}a \int_0^a \mathrm{d}b\sqrt{ab}\left(1-\frac{b}{a}\right) \mathrm{e}^{-2(a^2+b^2)}$$
$$= \mathcal{N}\int_0^\infty \mathrm{d}a \int_0^a \mathrm{d}b\sqrt{ab}\, \mathrm{e}^{-2(a^2+b^2)} - \mathcal{N}\int_0^\infty \mathrm{d}a \int_0^a \mathrm{d}b\sqrt{\frac{b^3}{a}}\, \mathrm{e}^{-2(a^2+b^2)}.$$

For the modulus we, therefore, get the estimate

$$|\langle d(1,2)\rangle| \leq \mathcal{N}\int_0^\infty \mathrm{d}a\sqrt{a}\, \mathrm{e}^{-2a^2} \int_0^\infty \mathrm{d}b\sqrt{b}\, \mathrm{e}^{-2b^2}$$
$$+ \mathcal{N}\int_0^\infty \mathrm{d}a \frac{1}{\sqrt{a}}\, \mathrm{e}^{-2a^2} \int_0^\infty \mathrm{d}b\sqrt{b^3}\mathrm{e}^{-2b^2} < \infty$$

which is due to the fact that

$$\int_0^\infty x^n \, e^{-kx^2} \, dx < \infty \quad \text{for} \quad n > -1.$$

This shows finiteness of $\langle d(1,2)\rangle$.

Example 4.7.

$$q = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}.$$

Here the most general Dirac operator is

$$D = \begin{pmatrix} 0 & 0 & m_1 & \bar{m}_1 \\ 0 & 0 & m_2 & \bar{m}_2 \\ \bar{m}_1 & \bar{m}_2 & 0 & 0 \\ m_1 & m_2 & 0 & 0 \end{pmatrix}, \quad m_1, m_2 \in \mathbb{C}$$

acting on $\mathcal{H} = \mathcal{H}_{11} \oplus \mathcal{H}_{12} \oplus \mathcal{H}_{21} \cong \mathbb{C}^2 \oplus \mathbb{C} \oplus \mathbb{C}$ and admissible unitary transformations are

$$U = \begin{pmatrix} R & 0 & 0 \\ 0 & e^{i\varphi} & 0 \\ 0 & 0 & e^{-i\varphi} \end{pmatrix}, \quad R \in O(2), \ \ e^{i\varphi} \in U(1).$$

If we put $\vec{m} := \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$, then the Dirac operator can be written as

$$D = \begin{pmatrix} 0 & \vec{m} & \bar{\vec{m}} \\ \vec{m}^* & 0 & 0 \\ \vec{m}^{\mathrm{T}} & 0 & 0 \end{pmatrix}$$

and unitary transformations lead to

$$\begin{aligned} UDU^* &= \begin{pmatrix} R & 0 & 0 \\ 0 & e^{i\varphi} & 0 \\ 0 & 0 & e^{-i\varphi} \end{pmatrix} \begin{pmatrix} 0 & \vec{m} & \bar{\vec{m}} \\ \vec{m}^* & 0 & 0 \\ \vec{m}^{\mathrm{T}} & 0 & 0 \end{pmatrix} \begin{pmatrix} R^{\mathrm{T}} & 0 & 0 \\ 0 & e^{-i\varphi} & 0 \\ 0 & 0 & e^{i\varphi} \end{pmatrix} \\ &= \begin{pmatrix} 0 & R\vec{m}\, e^{-i\varphi} & R\bar{\vec{m}}\, e^{i\varphi} \\ e^{i\varphi}\vec{m}^* R^{\mathrm{T}} & 0 & 0 \\ e^{-i\varphi}\vec{m}^{\mathrm{T}} R^{\mathrm{T}} & 0 & 0 \end{pmatrix}. \end{aligned}$$

Again one can find representatives of the equivalence classes using the following theorem.

Theorem 4.8. *Let* $\vec{m} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \in \mathbb{C}^2$. *Then there exist* $R \in O(2)$, $\varphi \in [0, 2\pi[$ *and a unique* $\psi \in [0, \pi/2]$ *such that*

$$R\vec{m}\,\mathrm{e}^{\mathrm{i}\varphi} = \frac{|\vec{m}|}{\sqrt{2}} \begin{pmatrix} 1 \\ \mathrm{e}^{\mathrm{i}\psi} \end{pmatrix}. \tag{9}$$

So Dirac operators are parametrized by vectors of the form $\rho/\sqrt{2} \begin{pmatrix} 1 \\ \mathrm{e}^{\mathrm{i}\psi} \end{pmatrix}$, $\rho \in [0, \infty[$, $\psi \in [0, \pi/2]$, *i.e. a "quarter of a cone" in* $\mathbb{C}^2$.

The proof is similar to Theorem 4.6, so we skip it here.

The eigenvalues of D in this parametrization are given by

$$\lambda_+^2 = \tfrac{1}{2}\rho^2(2 + \sqrt{2 + 2\cos(2\psi)}), \tag{10}$$

$$\lambda_-^2 = \tfrac{1}{2}\rho^2(2 - \sqrt{2 + 2\cos(2\psi)}). \tag{11}$$

So for

$$S(D) = \tfrac{1}{4}t\,\mathrm{tr}\,D^2 = t\rho^2,$$

one can for example calculate

$$\langle \lambda_+^2 - \lambda_-^2 \rangle = \langle \rho^2 \sqrt{2 + 2\cos(2\psi)} \rangle \tag{12}$$

with the (obvious) measure

$$\int \mathrm{d}\rho \int \mathrm{d}\psi. \tag{13}$$

The only thing we need to take care of is the interval of integration for the variable ψ which is $[0, \pi/2]$ due to Theorem 4.8. So for this case we get

$$\int_0^\infty \mathrm{d}\rho \int_0^{\pi/2} \mathrm{d}\psi \rho^2 \sqrt{2 + 2\cos(2\psi)}\,\mathrm{e}^{-t\rho^2} = \sqrt{\frac{\pi}{4t^3}} > 0. \tag{14}$$

Thus, also in this example we observe the effect of spontaneously broken invariance. Unlike the previous example, the measure we used here is, however, spectral invariant:

Remember that (if there are four independent eigenvalues of D) a spectral invariant measure must be of the form

$$f(\lambda_1, \lambda_2, \lambda_3, \lambda_4)\,\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2\,\mathrm{d}\lambda_3\,\mathrm{d}\lambda_4,$$

where f is totally symmetric with respect to permutations of its arguments. In our case the eigenvalues of the Dirac operator are not independent: $\lambda_1 = -\lambda_2$, $\lambda_3 = -\lambda_4$ so one can as well choose the invariant measure as

$$f(\lambda_1, \lambda_3)\,\mathrm{d}\lambda_1\,\mathrm{d}\lambda_3$$

(in the integral λ_1 and λ_3 are now running from 0 to ∞). To show invariance of (13) we first prove the following lemma.

Lemma 4.9. *If the function f has the properties $|f(\lambda_+,\lambda_-)| = |f(\lambda_-,\lambda_+)|\ \forall\lambda_+,\lambda_-$ and $f(\lambda_+,\lambda_-) \geq 0$ if $\lambda_+ \geq \lambda_-$ then the expression $\int_0^\infty \mathrm{d}\lambda_+ \int_0^{\lambda_+} \mathrm{d}\lambda_-\, f(\lambda_+,\lambda_-)$ is spectral invariant.*

Proof. According to the assumption (the symmetry of $|f|$) the expression

$$\int_0^\infty \mathrm{d}\lambda_+ \int_0^\infty \mathrm{d}\lambda_- |f(\lambda_+,\lambda_-)|$$

is spectral invariant. If one splits the domain of integration as follows:

$$\int_0^\infty \mathrm{d}\lambda_+ \int_0^\infty \mathrm{d}\lambda_- = \int_0^\infty \mathrm{d}\lambda_+ \int_0^{\lambda_+} \mathrm{d}\lambda_- + \int_0^\infty \mathrm{d}\lambda_- \int_0^{\lambda_-} \mathrm{d}\lambda_+,$$

one gets

$$\begin{aligned}&\int_0^\infty \mathrm{d}\lambda_+ \int_0^\infty \mathrm{d}\lambda_- |f(\lambda_+,\lambda_-)|\\ &\quad= \int_0^\infty \mathrm{d}\lambda_+ \int_0^{\lambda_+} \mathrm{d}\lambda_- |f(\lambda_+,\lambda_-)| + \int_0^\infty \mathrm{d}\lambda_- \int_0^{\lambda_-} \mathrm{d}\lambda_+ |f(\lambda_+,\lambda_-)|.\end{aligned}$$

After renaming $\lambda_+ \leftrightarrow \lambda_-$ and using the symmetry of $|f|$ one gets equality of the two terms on the r.h.s. and finally

$$\int_0^\infty \mathrm{d}\lambda_+ \int_0^\infty \mathrm{d}\lambda_- |f(\lambda_+,\lambda_-) = 2\int_0^\infty \mathrm{d}\lambda_+ \int_0^{\lambda_+} \mathrm{d}\lambda_- |f(\lambda_+,\lambda_-).$$

Now we can use the property $|f(\lambda_+,\lambda_-)| = f(\lambda_+,\lambda_-)\ \forall\lambda_+ \geq \lambda_-$ to see that

$$\int_0^\infty \mathrm{d}\lambda_+ \int_0^{\lambda_+} \mathrm{d}\lambda_-\, f(\lambda_+,\lambda_-) = \frac{1}{2}\int_0^\infty \mathrm{d}\lambda_+ \int_0^\infty \mathrm{d}\lambda_- |f(\lambda_+,\lambda_-)|$$

is indeed spectral invariant. □

Let us now check, whether the measure (13) fulfills the requirements of Lemma 4.9. Summing (10) and (11) leads to

$$\lambda_+^2 + \lambda_-^2 = 2\rho^2 \Rightarrow \rho = \sqrt{\tfrac{1}{2}(\lambda_+^2 + \lambda_-^2)}.$$

Taking the difference of the two expressions gives

$$\begin{aligned}\lambda_+^2 - \lambda_-^2 &= \rho^2\sqrt{2 + 2\cos(2\psi)} = \frac{1}{2}(\lambda_+^2 + \lambda_-^2)\sqrt{2 + 2\cos(2\psi)}\\ &\Rightarrow \cos(2\psi) = 2\left(\frac{\lambda_+^2 - \lambda_-^2}{\lambda_+^2 + \lambda_-^2}\right)^2 - 1\\ &\Rightarrow \psi = \frac{1}{2}\arccos\left(2\left(\frac{\lambda_+^2 - \lambda_-^2}{\lambda_+^2 + \lambda_-^2}\right)^2 - 1\right).\end{aligned}$$

In the following we put $x := \lambda_+$, $y := \lambda_-$ to alleviate notation. So, we have

$$\rho = \sqrt{\frac{1}{2}(x^2 + y^2)} \qquad \psi = \frac{1}{2}\arccos\left(2\left(\frac{x^2 - y^2}{x^2 + y^2}\right)^2 - 1\right).$$

The measure transforms according to

$$\mathrm{d}\rho = \frac{\partial\rho}{\partial x}\mathrm{d}x + \frac{\partial\rho}{\partial y}\mathrm{d}y,$$
$$\mathrm{d}\psi = \frac{\partial\psi}{\partial x}\mathrm{d}x + \frac{\partial\psi}{\partial y}\mathrm{d}y \Rightarrow \mathrm{d}\rho\,\mathrm{d}\psi = \underbrace{\left(\frac{\partial\rho}{\partial x}\frac{\partial\psi}{\partial y} - \frac{\partial\rho}{\partial y}\frac{\partial\psi}{\partial x}\right)}_{=:J(x,y)}\mathrm{d}x\,\mathrm{d}y.$$

Calculating J leads to

$$J(x, y) = \frac{4xy(x^2 - y^2)}{(x^2 + y^2)^2\sqrt{2(x^2 - y^2)^2/(x^2 + y^2)[1 - (x^2 - y^2)/(x^2 + y^2)^2]}}$$
$$= \frac{\sqrt{2}}{\sqrt{x^2 + y^2}}\mathrm{sign}(x^2 - y^2).$$

As one can see, J has the properties that are postulated in Lemma 4.9 and so the integral

$$\int_0^\infty \mathrm{d}\rho \int_0^{\pi/2} \mathrm{d}\psi = \int_0^\infty \mathrm{d}\lambda_+ \int_0^{\lambda_+} \mathrm{d}\lambda_- J(\lambda_+, \lambda_-)$$

is spectral invariant. Thus there is a less obvious loss of spectral invariance in the transition from classical to quantized theory in this example.

5. Discussion/outlook

In this paper we entered the subject of quantizing finite-dimensional spectral triples only superficially. In fact, we mentioned only few points concerning only two points. However, from such a baby toy model one should not expect more than an incomplete illustration.

In particular, the reader might have missed the usual folkloristic results about the Planck length, i.e. a minimal measurable distance of (the) two points. We have, actually, not been able to obtain such results in our models. For the simplest Example 4.2, when D has only one eigenvalue, and for the Gaussian measure, the vacuum expectation value of the distance is given by

$$\langle 0|d(1, 2)|0\rangle = \mathcal{N}\int_0^\infty \mathrm{d}\lambda \frac{1}{\lambda}\mathrm{e}^{-t\lambda^2}.$$

For an arbitrary state

$$|f\rangle = f(\lambda)|0\rangle, \qquad \langle f|f\rangle = \mathcal{N}\int_0^\infty \mathrm{d}\lambda |f|^2\,\mathrm{e}^{-t\lambda^2} = 1$$

with some suitable function f, one then easily verifies that the expectation value

$$\langle f|d(1,2)|f\rangle = \mathcal{N}\int_0^\infty \mathrm{d}\lambda \frac{|f|^2}{\lambda}\mathrm{e}^{-t\lambda^2}$$

is only bounded from below by zero: Consider, for instance, $|f(\lambda)|^2 = \mathrm{e}^{t\lambda_0^2}\delta(\lambda-\lambda_0)/\mathcal{N}$, in which case $\langle f|d(1,2)|f\rangle = 1/\lambda_0$.

However, such arguments do not appear too compelling (to us) in view of the severe problem of lacking time. As mentioned in Section 1, models based on algebras

$$\mathcal{A} = C_0^\infty(\mathbb{R}) \otimes \mathcal{A}_F,$$

where $\mathcal{A}_F$ denotes any finite-dimensional C^*-algebra, while $C_0^\infty(\mathbb{R})$ is interpreted as functions on the time axis, are currently under construction. It is our ultimate aim in this project, to investigate models of this type. It is then possible to meaningfully define classical, canonical transformations—under which a diffeomorphism invariant measure for the quantization could be invariant—and so on. Even more so, one can then dream of approximating spacetime by such a model, or more precisely: space by $\mathcal{A}_F$. But all that is music of the future (but not of the past).

That does not mean, however that it is in vain to study models based on finite-dimensional spectral triples. First of all, solving the technical problems they pose, is an unavoidable preparation before dealing with the more complicated models where time is included. Secondly, though the "physical" interpretation of these models is not clear from the start, it is also not clear that it doesn't exist. Moreover these systems show some remarkable effects, for which we would like to gain some intuition. For example, we have seen that the vacuum expectation value of the distance of the two points is infinite when D has only one independent eigenvalue, but comes out being finite if there are more eigenvalues of D. In the example we presented, this seems to be related to the spontaneous breaking of spectral invariance, but in other examples this is not the case. In a forthcoming paper we shall for instance present an example, where some "distance-like" observable is given as $d = 1/\max\{\lambda_1, \lambda_2\}$. Here λ_1, λ_2 denote the two independent eigenvalues of D—if there were only one eigenvalue, d would equal the inverse of this value—and the expectation value is given as

$$\langle d\rangle = \mathcal{N}\int_0^\infty\int_0^\infty \mathrm{d}\lambda_1\mathrm{d}\lambda_2 \frac{1}{\max\{\lambda_1,\lambda_2\}}\mathrm{e}^{-t(\lambda_1^2+\lambda_2^2)}.$$

It is a nice exercise to compute this *finite* expectation value [11].

Last not least, as they lead to rather unusual matrix models, the quantization of finite-dimensional spectral triples might be interesting enough by itself. In this respect, it is particularly challenging to study the various possible continuum limits these systems offer.

Most obviously one could consider N-point spaces and eventually the continuum-limit $N\to\infty$ which corresponds to a lattice. Secondly, one might study proper noncommutative examples, or one can consider limits $n_i\to\infty$ in the decomposition of the algebra

$$A = \oplus_{i=1}^N M_{n_i}(\mathbb{C}).$$

But one could also study limits where some or all entries of the intersection form q are sent to infinity: In the case of the 2-point space this would (as a first step) require the classification of equivalent Dirac operators for the relation

$$(M, N) \sim (AMV^*, VNB^{\mathrm{T}}),$$

where $M : \mathbb{C}^l \to \mathbb{C}^k$, $N : \mathbb{C}^m \to \mathbb{C}^l$ denote the independent blocks of the Dirac operator (as in Section 3) and $A \in O(k)$, $V \in U(l)$ and $B \in O(m)$ parametrize the unitary equivalence. A general solution (i.e. for arbitrary k, l, m) is not yet found.

All that, of course, requires in addition a much more systematic construction of path integrals for (as generic as possible) finite-dimensional spectral triples, and this will therefore be the subject of our subsequent paper.

Appendix A. A proof of the canonical form of J

As the proofs in the literature of the canonical form of J that we have stated in Section 2 are not completely compelling, we would like to present here another proof of this (nevertheless correct) statement.

Before giving our new proof we shall briefly describe at which point it essentially improves on the literature.

As an antiunitary operator J is of the form $J = KU$, where K denotes the antilinear operator of complex conjugation, while U is unitary. All the proofs in the literature use the fact that U can be diagonalized. Exploiting the antilinearity of J, it is then rather easy to get rid of the eigenvalues of U by a suitable redefinition of the basis.

It is however not shown that one can also diagonalize the combination KU, i.e. that there exists a unitary matrix R such that

$$RKUR^* = K\bar{R}UR^* = KU_d$$

where U_d is diagonal. We shall prove that one can in fact find a real (orthogonal) matrix R doing the job.

Lemma A.1. *There always exists an othornormal basis $\{v_1, \ldots, v_n\}$ of $\mathcal{H}_{kl}$ and $\{w_1, \ldots, w_n\}$ of $\mathcal{H}_{lk}$, respectively, such that for $k \neq l$ one has*

$$Jv_i = w_i,$$

and for $k = l$ (with $v_i = x \otimes y \otimes z$) it is

$$Jv_i = J(x \otimes y \otimes z) = \bar{z} \otimes \bar{y} \otimes \bar{x}. \tag{A.1}$$

Thus J essentially interchanges the basis vectors of the different subspaces of $\mathcal{H}$

Proof. First of all note that J defined by

$$J(u \otimes v \otimes w) = \bar{w} \otimes \bar{v} \otimes \bar{u} \tag{A.2}$$

fulfills the axioms $J^2 = 1$ and $[\pi(a), J\pi(b)J] = 0$ (checking this is straightforward). Now, let $\tilde{J}$ be another antiunitary map with the properties $\tilde{J}^2 = 1$ and $\pi^o(a) = \tilde{J}\pi(a^*)\tilde{J}$. Then $J \circ \tilde{J}$ is an invertible map that commutes with both representations of $\mathcal{A}$,

$$[J \circ \tilde{J}, \pi(a)] = [J \circ \tilde{J}, \pi^o(a)] = 0,$$

since

$$J \circ \tilde{J}\pi(a) = J \circ \tilde{J}\pi(a)\tilde{J}^2 = J \circ \pi^o(a^*) \circ \tilde{J} = J \circ J \circ \pi(a) \circ J \circ \tilde{J} = \pi(a) J \circ \tilde{J},$$

and analogously for π^o. Thus $J \circ \tilde{J}$ is of the form

$$J \circ \tilde{J} = \mathbf{1} \otimes j \otimes \mathbf{1}. \tag{A.3}$$

As $J \circ \tilde{J}$ is unitary

$$\langle \psi, \phi \rangle = \langle \tilde{J}\phi, \tilde{J}\psi \rangle = \langle J \circ \tilde{J}\psi, J \circ \tilde{J}\phi \rangle,$$

also j is unitary. Consider the following two cases: □

Case 1 ($k \neq l$). Choose an arbitrary basis $\{v_1, \ldots, v_n\}$ in $\mathbb{C}^{r_{kl}}$ and define the basis in $\mathbb{C}^{r_{lk}}$ through

$$w_i := J \circ (1 \otimes j \otimes 1) v_i.$$

Since $\tilde{J} = J \circ (1 \otimes j \otimes 1)$ it then immediately follows that $w_i = \tilde{J} v_i$ holds.

Case 2 ($k = l$). Note that $J^2 = \tilde{J}^2 = 1$, which implies $\bar{j} j = 1$

$$\begin{aligned}
(A.3) \Rightarrow \tilde{J} &= J \circ (1 \otimes j \otimes 1) \\
\Rightarrow \tilde{J}^2 &= J \circ (1 \otimes j \otimes 1) \circ J \circ (1 \otimes j \otimes 1) \\
&= (1 \otimes \bar{j} \otimes 1) \circ J^2 \circ (1 \otimes j \otimes 1) \quad \text{according to (A.2)} \\
&= (1 \otimes \bar{j} j \otimes 1).
\end{aligned}$$

Thus j fulfills the relations $j^* = j^{-1}$ and $\bar{j} = j^{-1}$. To such a matrix j there always exists a unitary matrix u with the property

$$j = u^{\mathrm{T}} u. \tag{A.4}$$

Proof of formula (A.4). Consider $j \in M_n(\mathbb{C})$. We prove the statement by induction on n.

The case $n = 1$ is trivial: For $j = \mathrm{e}^{\mathrm{i}\varphi}$ one can simply take $u = \mathrm{e}^{\mathrm{i}\varphi/2}$.

Suppose now the statement has been proven for all $k = 1, \ldots, n$ and let $j \in M_{n+1}(\mathbb{C})$. Since j is unitary, its eigenvalues $\lambda_1, \ldots, \lambda_{n+1}$ are complex with modulus 1. Since $\bar{j} = j^{-1}$ the following is true.

If ψ is an eigenvector of j for the eigenvalue λ, then also its complex conjugate vector $\bar{\psi}$ is an eigenvector for the same eigenvalue λ. To see this, note that a matrix and its inverse do have the same eigenvectors, and thus also j and $\bar{j}$ do have the same eigenvectors (which is not true in general). Hence one has the equation (recall $\lambda^{-1} = \bar{\lambda}$)

$$\bar{j}\psi = \bar{\lambda}\psi$$

which after complex conjugation shows that $\bar{\psi}$ is an eigenvector of j for the eigenvalue λ.

Now if j had only one eigenvalue λ, being $(n+1)$-times degenerate, it would be proportional to the identity $j = \lambda\mathbf{1}$. So in such a case the existence of u with $j = u^{\mathrm{T}}u$ would be obvious.

Accordingly we can suppose in the following that j has at least two different eigenvalues. Let λ be one of these eigenvalues, E_λ be the corresponding eigenspace. By assumption its dimension k can at most equal n. Let $\{\psi_1, \dots, \psi_k\}$ denote an orthonormal basis of E_λ. Then also the complex conjugate vectors belong to E_λ and one easily verifies that $\{\bar{\psi}_1, \dots, \bar{\psi}_k\}$ is another orthonormal basis of E_λ. The interchange between these two (possibly different) bases is described by a unitary matrix a:

$$\bar{\psi}_i = a_{ji}\psi_j \tag{A.5}$$

(with summation over the index j). But then by complex conjugation of (A.5) it follows that $\psi_i = \bar{a}_{ji}\bar{\psi}_j$ and hence $\psi_j = \bar{a}_{kj}\bar{\psi}_k$. Plugging this expression into (A.5), we conclude

$$\bar{\psi}_i = a_{ji}\bar{a}_{kj}\bar{\psi}_k \Rightarrow \bar{a}_{kj}a_{ji} = \delta_{ki} \quad (\text{since } \{\bar{\psi}_i\} \text{ is ONB}) \Rightarrow \bar{a} = a^{-1}.$$

Conclusively the matrix a fulfills the hypothesis of the induction and hence it can be written as

$$a = bb^{\mathrm{T}} \quad \Leftrightarrow \quad b_{ij}b_{kj} = a_{ik}.$$

If we now define a new ONB of E_λ through

$$\phi_i := b_{ji}\psi_j$$

then it follows:

$$\bar{\phi}_i = \bar{b}_{ji}\bar{\psi}_j = \bar{b}_{ji}a_{kj}\psi_k = \bar{b}_{ji}b_{kl}b_{jl}\psi_k = b_{kl}\underbrace{b^*_{ij}b_{jl}}_{\delta_{il}}\psi_k = b_{ki}\psi_k = \phi_i$$

Thus $\{\phi_1, \dots, \phi_k\}$ is a *real* orthonormal basis of E_λ. Since this is true for any eigenspace of j, we conclude that one can find a complete set of real, orthonormal eigenvectors of j. Hence j can be diagonalized by a (real) orthogonal matrix R:

$$RjR^{\mathrm{T}} = j_d = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\rho_1}, \dots, \mathrm{e}^{\mathrm{i}\rho_{n+1}}).$$

Set $u = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\rho_1/2}, \dots, \mathrm{e}^{\mathrm{i}\rho_{n+1}/2})R$. Then one checks $j = u^{\mathrm{T}}u$ and this proves (A.4).

As a consequence of the decomposition $j = u^{\mathrm{T}}u$ it then follows that:

$$\begin{aligned}(1 \otimes u \otimes 1) \circ \tilde{J} \circ (1 \otimes u^* \otimes 1) &= (1 \otimes u \otimes 1) \circ J \circ (1 \otimes j \otimes 1) \circ (1 \otimes u^* \otimes 1)\\ &= J \circ (1 \otimes \bar{u}ju^* \otimes 1).\end{aligned}$$

But $\bar{u}ju^* = \bar{u}u^{\mathrm{T}}uu^* = 1$, and therefore

$$(1 \otimes u \otimes 1) \circ \tilde{J} \circ (1 \otimes u^* \otimes 1) = J.$$

So changing the basis with the help of u will cast $\tilde{J}$ in the desired form. □

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